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The dispersion relation for a nonlinear random gravity wave field

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The dispersion relation for a random gravity wave field is derived using the complete system of nonlinear equations. It is found that the generally accepted dispersion relation is only a first-order approximation to the mean value. The correction to this approximation is expressed in terms of the energy spectral function of the wave field. The non-zero mean deviation is proportional to the ratio of the mean Eulerian velocity at the surface and the local phase velocity. In addition to the mean deviation, there is a random scatter. The root-mean-square value of this scatter is proportional to the ratio of the root-mean-square surface velocity and the local phase velocity. As for the phase velocity, the non-zero mean deviation is equal to the mean Eulerian velocity while the root-mean-square scatter is equal to the root-mean-square surface velocity. Special cases are considered and a comparison with experimental data is also discussed.

1. Introduction

In the study of surface waves, there is a special relationship between the wave-number k and the frequency σ, known as the dispersion relation. That is, in deep water,

$$\sigma^2 = gk, \tag{1}$$

where g is the gravitational acceleration. Although this relation is derived for a single wave and based on linear theory only, it has been used as an approximation even in random wave fields (see, for example, Phillips 1966). On close examination of the higher-order corrections, Stokes (1847) calculated that to the third order of approximation

$$\sigma^2 = gk(1+a^2k^2+O(a^4k^4)), \tag{2}$$

or

$$C^2 = gk^{-1}(1+a^2k^2+O(a^4k^4)), \tag{3}$$

in which a is the wave amplitude and C is the phase velocity. Thus (owing to nonlinear effects) the dispersion relation and phase speed of the waves change. In fact, the experimental studies by Grose, Warsh & Garstang (1972), Yefimov & Khristoforov (1971) and Yefimov, Solov'yev & Khristoforov (1972) all indicate qualitative agreement with Stokes' result, but the data showed a great deal of

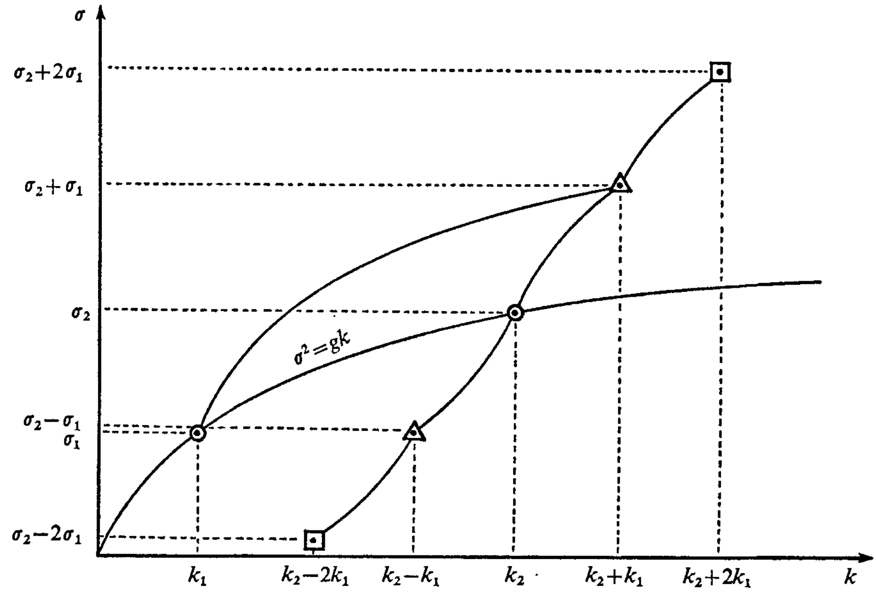

FIGURE 1. Higher-order waves generated by nonlinear interactions between two primary wave trains. ○, dispersion relation of the primary waves, of frequencies σ_1, σ_2 and wavenumbers k_1, k_2; △, second-generation waves, of frequencies $\sigma_1 \pm \sigma_2$ and wavenumbers $k_1 \pm k_2$; □, third-generation waves, of frequencies $\sigma_1 \pm 2\sigma_2$ and wavenumbers $k_1 \pm 2k_2$.

scatter. The effect of the nonlinear mechanism can be easily demonstrated by the following simple discussion.

Let us take two trains of simple waves propagating in the same direction with wavenumbers k_1 and k_2 and frequencies σ_1 and σ_2, respectively. For each of the wave trains, the following simple dispersion relation holds:

$$\sigma_i^2 = gk_i, \quad \text{where} \quad i = 1, 2. \tag{4}$$

However, owing to nonlinear interaction, waves of wavenumbers $k_1 \pm k_2$ and frequencies $\sigma_1 \pm \sigma_2$ will be generated as the first-generation offspring of the original waves. It is noted that the dispersion relation of these waves is not $(\sigma_1 \pm \sigma_2)^2 = g(k_1 \pm k_2)$. The nonlinear mechanism will not stop here. Further interactions will generate waves of wavenumbers and frequencies $k_1 \pm 2k_2$, $\sigma_1 \pm 2\sigma_2$, etc. as discussed by Phillips (1960*a*), Longuet-Higgins (1962) and Longuet-Higgins & Phillips (1962). The dispersion relation of these waves again does not satisfy $(\sigma_1 \pm 2\sigma_2)^2 = g(k_1 \pm 2k_2)$. The whole process of this interaction is shown schematically in figure 1. For a random wave field, the analogous process will involve all the components and eventually spread the energy over wavenumber–frequency space according to the nonlinear equations of motion. The dispersion relation can no longer be represented by a single line but will be modified accordingly.

Owing to the special significance of the dispersion relation in wave studies, a detailed understanding of it will have important consequences on such problems as random wave interactions and transformation between wavenumber and frequency spectra. In this paper, a quantitative analysis based on the nonlinear kinematic and dynamic equations is performed to derive the most general form of the dispersion relation for any given homogeneous random gravity wave field. The mechanism of nonlinear wave–wave interactions as studied by Phillips (1960*a*, 1966) is fully incorporated. The procedure is exactly the same as in the simple wave case, but random wave representation is used. Special cases are subsequently discussed. It is shown that the dispersion relation is random; the mean dispersion relation deviates from the linear case and the mean-square random scatter is also obtained. It is noted that the mean-square random scatter is similar to that obtained by Longuet-Higgins & Phillips (1962) but that the latter considered only the case of discrete components under resonant interaction conditions. Furthermore, Willebrand (1975), using a variational principle, showed that in an inhomogeneous wave field the group velocity or energy transport is influenced by nonlinear interactions; in a homogeneous random wave field, although nonlinear effects do not contribute to the total energy transport, they do modify both the mean value and random scatter of the dispersion relation.

2. Analysis

The following analysis is based mainly on the discussion of random waves by Phillips (1966, pp. 27–79). The special approach adopted follows that of Huang (1971). Under the standard assumption of waves of small slope on an inviscid and incompressible fluid, the whole fluid field can be approximated by an irrotational motion governed by

$$\nabla^2\Phi(\mathbf{x}, z, t) = 0, \tag{5}$$

where $\Phi(\mathbf{x}, z, t)$ is the velocity potential, $\mathbf{x}$ the horizontal position vector, z the vertical position, measured positive upwards, and t the time. For a random wave field, the solution of (5) subject to the condition that all motion ceases at infinite depth is

$$\Phi(\mathbf{x}, z, t) = \int_{\mathbf{k}}\int_{n} dA(\mathbf{k}, n)\, e^{|\mathbf{k}|z}\, e^{i(\mathbf{k}\cdot\mathbf{x}-nt)}, \tag{6}$$

where $dA(\mathbf{k}, n)$ is a complex-valued random function of the wavenumber $\mathbf{k}$ and frequency n. The integrals are carried over all wavenumber–frequency space.

For this wave field, the free-surface elevation $\zeta(\mathbf{x}, t)$ can be expressed as

$$\zeta(\mathbf{x}, t) = \int_{\mathbf{k}}\int_{n} dB(\mathbf{k}, n)\, e^{i(\mathbf{k}\cdot\mathbf{x}-nt)}, \tag{7}$$

where $dB(\mathbf{k}, n)$ is another complex-valued random function. Furthermore, under statistically stationary and homogeneous assumptions, $dB(\mathbf{k}, n)$ can be related to the directional wave energy spectrum $X(\mathbf{k}, n)$ by

$$\overline{dB(\mathbf{k}, n)\, dB^*(\mathbf{k}_1, n_1)} = \begin{cases} 0 & \text{if} \quad \mathbf{k} \neq \mathbf{k}_1, \quad n \neq n_1, \\ X(\mathbf{k}, n) & \text{if} \quad \mathbf{k} = \mathbf{k}_1, \quad n = n_1, \end{cases} \tag{8}$$

where $dB^*(\mathbf{k}, n)$ is the complex conjugate of $dB(\mathbf{k}, n)$ and the overbar denotes an ensemble average. Thus the sea state is uniquely statistically related to the

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random function $dB(\mathbf{k}, n)$, which in turn determines the spectral function. Since the wave field is most commonly represented by the wave spectrum, it is desirable to relate other quantities to $dB(\mathbf{k}, n)$ and ultimately express all quantities in terms of the spectral function. To achieve this a relationship between $dA(\mathbf{k}, n)$ and $dB(\mathbf{k}, n)$ was found by Huang (1971) through the kinematic boundary condition

$$\frac{\partial \zeta}{\partial t} = \left.\frac{\partial \Phi}{\partial z}\right|_{\zeta} - (\nabla \Phi)_{\zeta} . \nabla \zeta. \tag{9}$$

To the third-order approximation, the result is

$$\begin{aligned} dA(\mathbf{k}, n) = &-\frac{in}{|\mathbf{k}|} dB(\mathbf{k}, n) + i \int_{\mathbf{k}_1} \int_{n_1} \frac{\mathbf{k} . (\mathbf{k} - \mathbf{k}_1)}{|\mathbf{k}|\,|\mathbf{k} - \mathbf{k}_1|} (n - n_1)\mathbf{k} \\ &\times dB(\mathbf{k} - \mathbf{k}_1, n - n_1)\, dB(k_1, n_1) \\ &+ i \int_{\mathbf{k}_1} \int_{n_1} \int_{\mathbf{k}_2} \int_{n_2} \frac{(n - n_1 - n_2)}{|\mathbf{k}|} \left\{ \tfrac{1}{2}(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) . (\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \right. \\ &\left. - \frac{\mathbf{k} . (\mathbf{k} - \mathbf{k}_1)\, (\mathbf{k} - \mathbf{k}_1) . (\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)}{|\mathbf{k} - \mathbf{k}_1|\,|\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2|} \right\} \\ &\times dB(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2, n - n_1 - n_2)\, dB(\mathbf{k}_1, n_1)\, dB(\mathbf{k}_2, n_2). \end{aligned} \tag{10}$$

Now by using the dynamic surface boundary condition without surface tension,

$$(\partial \Phi / \partial t)_{\zeta} + \tfrac{1}{2}(\mathbf{q}^2)_{\zeta} - g\zeta = 0, \tag{11}$$

with $\mathbf{q}$ as the velocity, an identity results. This is the same procedure as that used in deriving the dispersion relation for a train of simple waves. From (6), (7) and (11) we get

$$\begin{aligned} \int_{\mathbf{k}} \int_{n} g\, dB(\mathbf{k}, n)\, e^{i\chi} = &\int_{\mathbf{k}} \int_{n} in\, dA(\mathbf{k}, n)\, e^{|\mathbf{k}|\zeta} e^{i\chi} \\ &- \frac{1}{2} \int_{\mathbf{k}} \int_{n} \int_{\mathbf{k}_1} \int_{n_1} \{|\mathbf{k}|\,|\mathbf{k}_1| - \mathbf{k} . \mathbf{k}_1\}\, dA(\mathbf{k}, n)\, dA(\mathbf{k}_1, n_1) \\ &\times e^{(|\mathbf{k}| + |\mathbf{k}_1|)\zeta} e^{i(\chi + \chi_1)}, \end{aligned} \tag{12}$$

where $\chi = \mathbf{k} . \mathbf{x} - nt$ is the phase function. Substituting (10) into (12) and rearranging the terms, we obtain a relation of the form

$$\int_{\mathbf{k}} \int_{n} F(\mathbf{k}, n; \mathbf{k}_1, n_1, \mathbf{k}_2, n_2)\, dB(\mathbf{k}, n)\, e^{i\chi} \equiv 0. \tag{13}$$

Since $dB(\mathbf{k}, n)$ is an arbitrary function, the only possibility for (13) to hold identically is

$$F(\mathbf{k}, n; \mathbf{k}_1, n_1, \mathbf{k}_2, n_2) \equiv 0, \tag{14}$$

or

$$\begin{aligned} g - \frac{n^2}{|\mathbf{k}|} = &\int_{\mathbf{k}_1} \int_{n_1} f_1(\mathbf{k}, n; \mathbf{k}_1, n_1)\, dB(\mathbf{k}_1, n_1)\, e^{i\chi_1} \\ &- \int_{\mathbf{k}_1} \int_{n_1} \int_{\mathbf{k}_2} \int_{n_2} f_2(\mathbf{k}, n; \mathbf{k}_1, n_1, \mathbf{k}_2, n_2)\, dB(\mathbf{k}_1, n_1)\, dB(\mathbf{k}_2, n_2)\, e^{i(\chi_1 + \chi_2)} \\ &- \int_{\mathbf{k}_1} \int_{n_1} \int_{\mathbf{k}_2} \int_{n_2} f_3(\mathbf{k}, n; \mathbf{k}_1, n_1, \mathbf{k}_2, n_2)\, dB(\mathbf{k}_1 - \mathbf{k}_2, n_1 - n_2)\, dB(\mathbf{k}_2, n_2)\, e^{i\chi_1}, \end{aligned} \tag{15}$$

where

$$f_1 = n^2 - \frac{\mathbf{k}.(\mathbf{k}+\mathbf{k}_1)}{|\mathbf{k}|\,|\mathbf{k}+\mathbf{k}_1|}\, n(n+n_1) + \frac{1}{2}\left(1 - \frac{\mathbf{k}.\mathbf{k}_1}{|\mathbf{k}|\,|\mathbf{k}_1|}\right) nn_1,$$

$$f_2 = \frac{n(n+n_1+n_2)}{|\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2|}\left\{\tfrac{1}{2}(\mathbf{k}+2\mathbf{k}_1).\mathbf{k} - \frac{(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2).(\mathbf{k}+\mathbf{k}_2)\,(\mathbf{k}+\mathbf{k}_2).\mathbf{k}}{|\mathbf{k}+\mathbf{k}_2|\,|\mathbf{k}|}\right\}$$
$$- \frac{(\mathbf{k}+\mathbf{k}_1).\mathbf{k}}{|\mathbf{k}|}\, n(n+n_1) - \tfrac{1}{2}n^2|\mathbf{k}| - \frac{1}{2}\left(1 - \frac{\mathbf{k}.\mathbf{k}_1}{|\mathbf{k}|\,|\mathbf{k}_1|}\right)(|\mathbf{k}| + |\mathbf{k}_1|)\, nn_1$$
$$+ \frac{1}{2}\left(1 - \frac{(\mathbf{k}+\mathbf{k}_2).\mathbf{k}_1}{|\mathbf{k}+\mathbf{k}_2|\,|\mathbf{k}_1|}\right)\frac{(\mathbf{k}+\mathbf{k}_2).\mathbf{k}}{|\mathbf{k}|}\, nn_1,$$

$$f_3 = \left(1 - \frac{\mathbf{k}.\mathbf{k}_1}{|\mathbf{k}|\,|\mathbf{k}_1|}\right)\frac{\mathbf{k}_1.(\mathbf{k}_1-\mathbf{k}_2)}{|\mathbf{k}_1-\mathbf{k}_2|}\, n(n_1-n_2).$$

Equation (15) gives the most general expression for the dispersion relation up to the third-order terms. It indicates that in a random ocean the dispersion relation is also a function of time and space as expressed in the phase functions. Physically, this can be explained by considering the kinematics. When a short wave rides on a long wave, the short wave will experience a local Eulerian velocity which results in a Doppler frequency shift which is a function of time and space. However, (15) is too general to be of practical use without proper statistical processing. If we neglect all the nonlinear terms on the right-hand side of (15), we immediately recover the linear, first-order approximation

$$gk = n^2 = \sigma^2.$$

The nonlinear terms, which are generated by the interaction between different components in the random wave field, have a non-zero mean from the second and third terms on the right-hand side of (15), which denote the mean deviation of the dispersion relation from the result of linear theory, while the first term indicates random scatter with a zero mean. In order to calculate the mean deviation, and also the root-mean-square scatter, we have to perform the following calculations.

First, take the mean of (15). The first term on the right-hand side vanishes because it represents the random scatter only. The third term vanishes also because

$$\overline{dB(\mathbf{k}_1-\mathbf{k}_2, n_1-n_2)\, dB(\mathbf{k}_2, n_2)} = \overline{dB(\mathbf{k}_1-\mathbf{k}_2, n_1-n_2)\, dB^*(-\mathbf{k}_2, -n_2)}$$
$$= \begin{cases} 0 & \text{if } \mathbf{k}_1 \neq 0,\ n_1 \neq 0, \\ X(\mathbf{k}_2, n_2)\, d\mathbf{k}_2\, dn_2 & \text{if } \mathbf{k}_1 = n_1 = 0, \end{cases} \tag{16}$$

but when $\mathbf{k}_1 = n_1 = 0$ then $f_3 = 0$. The only non-zero term is the second term. After simplification, we have

$$g - \frac{n^2}{|\mathbf{k}|} = -\int_{\mathbf{k}_1}\int_{n_1} f_2'(\mathbf{k}, n;\, \mathbf{k}_1, n_1)\, X(\mathbf{k}_1, n_1)\, d\mathbf{k}_1\, dn_1, \tag{17}$$

where

$$f_2' = \frac{n^2}{|\mathbf{k}|}\left\{\tfrac{1}{2}(\mathbf{k}+2\mathbf{k}_1).\mathbf{k} - \frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1)\,(\mathbf{k}-\mathbf{k}_1).\mathbf{k}}{|\mathbf{k}|\,|\mathbf{k}-\mathbf{k}_1|}\right\} + \frac{(\mathbf{k}+\mathbf{k}_1).\mathbf{k}}{|\mathbf{k}|}\, n(n+n_1)$$
$$- \tfrac{1}{2}n^2|\mathbf{k}| - \frac{1}{2}\left(1 - \frac{\mathbf{k}.\mathbf{k}_1}{|\mathbf{k}|\,|\mathbf{k}_1|}\right)(|\mathbf{k}| + |\mathbf{k}_1|)\, nn_1 + \frac{1}{2}\left(1 - \frac{(\mathbf{k}-\mathbf{k}_1).\mathbf{k}_1}{|\mathbf{k}-\mathbf{k}_1|\,|\mathbf{k}_1|}\right)\frac{(\mathbf{k}-\mathbf{k}_1).\mathbf{k}}{|\mathbf{k}|}\, nn_1.$$

Thus we obtain the most general expression for the mean deviation in terms of the directional energy spectrum $X(\mathbf{k}, n)$. In order to simplify the expression further, let us assume that all the waves are propagating in the same direction. In this case

$$f_2' = 3n^2k_1 + nn_1(k+k_1) \tag{18}$$

and (17) becomes

$$g - \frac{n^2}{k} = -\int_{k_1}\int_{n_1} \{3n^2k_1 + nn_1(k+k_1)\} X(k_1, n_1)\, dk_1\, dn_1, \tag{19}$$

where $X(k_1, n_1)$ is the one-dimensional wavenumber–frequency spectrum. Since the spectral function $X(k, n)$ is an even function, we have simply that

$$g - \frac{n^2}{k} = -2n \int_{k_1}\int_{n_1} k_1 n_1 X(k_1, n_1)\, dk_1\, dn_1, \tag{20}$$

where the integrations are carried over positive values of k_1 and n_1 only.

Second, taking the mean of the square of the first term on the right-hand side of (15) we obtain the mean-square value $\overline{\epsilon^2}$ of the random scatter as

$$\overline{\epsilon^2} = \int_{\mathbf{k}_1}\int_{n_1} f_1^2(\mathbf{k}, n; \mathbf{k}_1, n_1)\, X(\mathbf{k}_1, n_1)\, d\mathbf{k}_1\, dn_1. \tag{21}$$

Under the assumption of a unidirectional wave field,

$$f_1^2(\mathbf{k}, n; \mathbf{k}_1, n_1) = n^2 n_1^2. \tag{22}$$

Hence

$$\overline{\epsilon^2} = 2n^2 \int_{k_1}\int_{n_1} n_1^2 X(k_1, n_1)\, dk_1\, dn_1, \tag{23}$$

where the integrations are again over positive k_1 and n only.

3. Phase velocity

Having derived the general expressions (20) and (23), we can calculate the changes in phase velocity in a random wave field. If we take the first-order approximation $g/k = C_0^2$, where C_0 is the first-order phase velocity, and by definition $C = n/k$, we can write (20) as

$$C^2 = \frac{n^2}{k^2} = C_0^2 \left\{1 + \frac{2C}{C_0} \int_{k_1}\int_{n_1} n_1 k_1 X(k_1, n_1)\, dk_1\, dn_1\right\}. \tag{24}$$

By using I to denote the integrand, (24) can be rewritten as

$$\left(\frac{C}{C_0}\right)^2 - \frac{2I}{C_0}\frac{C}{C_0} - 1 = 0. \tag{25}$$

The solution for $(I/C_0)^2 \ll 1$ is

$$\frac{C}{C_0} = \frac{I}{C_0} \pm \left\{\left(\frac{I}{C_0}\right)^2 + 1\right\}^{\frac{1}{2}} \simeq 1 + \frac{I}{C_0}, \tag{26}$$

or

$$C = C_0 + \int_{k_1}\int_{n_1} k_1 n_1 X(k_1, n_1)\, dk_1\, dn_1. \tag{27}$$

A similar result has been derived by Longuet-Higgins & Phillips (1962) by considering the interactions of two trains of simple waves propagating in the same direction. They found the change of phase velocity to be

$$\Delta C_2 = a_1^2 \sigma_1 k_2,$$

where the subscripts indicate the different wave trains. To generalize to a continuous spectral function, it was noted that

$$\sum_{\sigma}^{\sigma+d\sigma} \tfrac{1}{2} a_1^2 = E(\sigma)\, d\sigma, \tag{28}$$

in which $E(\sigma)$ is the wave energy spectrum. They then obtained

$$\Delta C_2 = 2\int_0^{\sigma_2} E(\sigma)\, k\sigma\, d\sigma + 2\int_{\sigma_2}^{\infty} E(\sigma)\, k_2 \sigma\, d\sigma. \tag{29}$$

The difference between the two expressions in (27) and (29) is a factor of two. Since the quantity represented by I is exactly the mean quasi-Eulerian velocity as derived by Phillips (1960 *b*), physically the change in phase velocity is precisely a Doppler shift caused by the local velocity field; therefore the present result should be expected.

4. Discussion

In order to show the effect of the nonlinear interaction on the dispersion relation in a random gravity wave field quantitatively, a specific spectral function has to be adopted. Since the dispersion relation depends on the spectral function for each specific sea state, an averaged form is not the natural expression to employ because spectral functions are different for different cases. However, an approximation can be made if the spectrum is written in the universal equilibrium form proposed by Phillips (1958):

$$X(k, n) = \beta g^2 \sigma^{-5} \delta(k - \sigma^2/g)\, \delta(n - \sigma), \tag{30}$$

where $\beta = 1{\cdot}2 \times 10^{-2}$ and δ is the Dirac delta function. Combining (20) and (30), we have in the mean

$$g - \sigma^2/k = -\beta g \sigma/\sigma_0, \tag{31}$$

with σ_0 as the cut-off frequency at the lower end.

The root-mean-square scatter can also be calculated by combining (23) and (30):

$$(\overline{\epsilon^2})^{\frac{1}{2}} = \pm \beta^{\frac{1}{2}} g\sigma/\sigma_0. \tag{32}$$

The scatter is much larger than the mean deviation since β is a small number. Physically, the scatter arises from Eulerian velocity components, which have small means but large fluctuations. A diagram showing the mean deviation and the root-mean-square scatter together with all the available data published by Longuet-Higgins, Cartwright & Smith (1963), Grose *et al.* (1972) and Yefimov *et al.* (1972) is given in figure 2. The agreement is in general rather poor. The general tendency seems to be a decrease in wavenumber in comparison with the

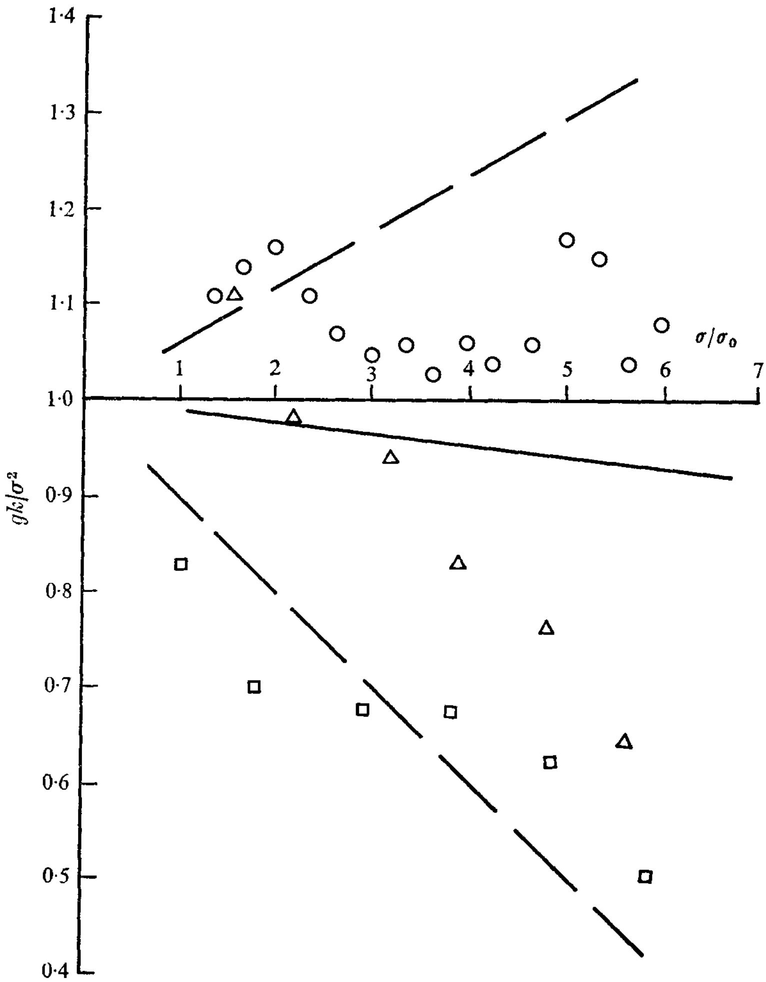

FIGURE 2. Comparison of theoretical dispersion relation for an equilibrium-range spectrum with some available field data. ——, mean deviation; – – –, root-mean-square scatter; ○, Longuet-Higgins *et al.* (1963); △, Grose *et al.* (1972); □, Yefimov *et al.* (1972).

corresponding linear waves. This apparent functional difference is especially evident in the higher frequency range, where the nonlinear mechanism is more important. Several reasons can be advanced to explain the discrepancy. First, the spectra in each individual case are not necessarily in the equilibrium range. Second, the wave fields are definitely not unidirectional. Third, as indicated by the low coherence measured by Yefimov *et al.* (1972), the wave field may contain motion other than waves, such as wind drift and turbulence. Finally, measurements were made in rather imprecise ways. Most of the wavenumber values reported were obtained from surface-slope data from wave staffs spaced at fixed distances. Such a set-up tends to underestimate the slope, and hence the value of the wavenumber, especially in the higher wavenumber range. For lack of better data, the diagram should be taken only as a general guide. The wide scatter serves as the best evidence of the absence of a single form of the dispersion relation for all wave conditions.

As the last example, let us consider the case of a single train of waves. The spectrum is given by

$$X(k,n) = \tfrac{1}{2}a^2\delta(k-k_0)\,\delta(n-n_0) \tag{33}$$

Combining (24) and (33), we have

$$C^2 = \sigma_0^2/k_0^2 = (g/k_0)\,(1+a^2k_0^2),$$

which is precisely the Stokes result. For this case the root-mean-square scatter is proportional to ak. This result compares well with the observed scatter value measured by Wright & Keller (1970) in their laboratory study.

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